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QED$_3$ Radiative Corrections in the Heisenberg Picture

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We evaluate the vacuum polarization tensor for three-dimensional quantum electrodynamics (QED$_3$) via Heisenberg equations of motion in order to clarify the problem arising from the use of different regularization prescriptions in the interaction picture. We conclude that the photon does acquire physical mass of topological origin when such contribution is taken into account for the photon propagator.

1. INTRODUCTION

Since the birth of quantum field theory, there has been an enormous amount of work done for the purpose of better understanding the divergences associated with higher-order computations in the perturbative scheme. Most of the work in this area has relied on the cleverly devised pictographical description that establishes a one-to-one correspondence between Feynman graphs and the terms of the time-ordered product expansion for the S-matrix "à la" Dyson. This intuitive approach due to Feynman allows one to quickly set up precise rules for the evaluation of the relevant contributions order by order in the perturbative series. There is no doubt that such a recipe has contributed much in helping physicists in the task of evaluating amplitudes and cross sections for different physical processes. Yet it becomes clear that this technique carries in itself a built-in characteristic inherent to the interaction picture (IP) from which it is derived.

In nonrelativistic quantum mechanics, where one deals with finite degrees of freedom, equivalent descriptions of a given system can be achieved in different "picture" representations. However, in the case of

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quantum field theory, where continuously infinite degrees of freedom are involved, one cannot in principle ascertain that such an equivalence persists (or even exists). Moreover, due to the characteristically divergent nature of the quantities associated with Feynman amplitudes, which calls for a convenient regularization prescription, that question of equivalence in different "picture" representations becomes somehow neglected (or even forgotten altogether) amid the hazy intricacies of perturbative calculations.

A few exceptions exist, though. In a series of papers Källén (1949) considered this issue in the context of mass and charge renormalization for QED in four dimensions. He tackled the problem using directly the field operators in the Heisenberg picture (HP), expanded in a power series in the coupling constant, and subsequently used them in the differential field equations of motion. He thus obtained an explicit expression for the vacuum polarization tensor considering separately the contributions to its real and imaginary parts, where only the real part carries a divergence, i.e., it needs to be regularized.

In a recent work we have considered radiative corrections in $(2+1)$-dimensional QED (Pimentel *et al.*, 1992)—which has not only theoretical interest on its own, but also becomes all the more interesting for having connections with high-temperature processes in four-dimensional theories—to check the generation of physical mass for the photon at the one-loop quantum level. This we did in the standard fashion by employing the usual Feynman rules of the IP, applying a gauge-invariant construct after analytical regularization to deal with ultraviolet singularities (Breitenlohner and Mitter, 1968). There we obtained a physical mass for the photon at the one-loop quantum level, in contrast to the Pauli–Villars regularization, where no such mass is generated.

Here we are going to use Källén's formalism (Källen, 1972) to evaluate the corresponding vacuum polarization tensor for QED_3, by imposing *a priori* conservation of charge, and confront the result with ours previously obtained (Pimentel *et al.*, 1992). Some illuminating insights emerging from the use of HP are discussed in our concluding remarks.

2. THE TRANSVERSE VACUUM POLARIZATION TENSOR

We can define the vacuum polarization tensor $\Pi_{\mu\nu}(k)$ in different ways. For example, let us consider an external field $\mathscr{A}_\mu^{\text{ext}}$ acting on the vacuum. There will be an induced current in the vacuum given by

$$\langle 0|j^\mu(x)|0\rangle = \int dx'\, e^2\Pi^{\mu\nu}(x-x')\mathscr{A}_\nu^{\text{ext}}(x') \tag{1}$$

The kernel $\Pi_{\mu\nu}(x-x')$ characterizes the linear response of the vacuum. We realize that the vacuum polarization tensor is the Fourier transform of this kernel. Thus, in three dimensions, we have

$$\Pi_{\mu\nu}(x - x') = \frac{1}{(2\pi)^3} \int d^3k\, e^{ik(x-x')} \Pi_{\mu\nu}(k) \tag{2}$$

The above definition will necessarily have the transverse properties. It follows from charge conservation that

$$\frac{\partial j_\mu(x)}{\partial x_\mu} = 0 \tag{3}$$

or, using equations (1) and (2),

$$k^\mu \Pi_{\mu\nu} = 0 \tag{4}$$

A covariant second-rank tensor $\Pi_{\mu\nu}(k)$ can be built from covariant objects like k_μ, $g_{\mu\nu}$ and $\epsilon_{\mu\nu\alpha}$ such that equation (4) is satisfied. In this sense, the vacuum polarization tensor ought to have the form

$$\Pi_{\mu\nu}(k) = G(k^2) k_\mu k_\nu + H(k^2) g_{\mu\nu} + im\epsilon_{\mu\nu\alpha} k^\alpha \Pi^{(2)}(k^2) \tag{5}$$

Unlike the four-dimensional case, $\Pi_{\mu\nu}(k)$ has an antisymmetric piece due to the Levi-Civita symbol $\epsilon_{\mu\nu\alpha}$. From equation (4) we have

$$\Pi_{\mu\nu}(k) = \Pi^{(1)}(k^2)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + im\epsilon_{\mu\nu\alpha} k^\alpha \Pi^{(2)}(k^2) \tag{6}$$

where we have defined

$$\Pi^{(1)}(k^2) \equiv -k^2 G(k^2) \tag{7}$$

If we contract $\Pi_{\mu\nu}(k)$ with $g^{\mu\nu}$ and $\epsilon^{\mu\nu\rho}$, we obtain, respectively,

$$\Pi^{(1)}(k^2) = \frac{1}{2} \Pi^\mu_\mu(k) \tag{8}$$

$$\Pi^{(2)}(k^2) = -\frac{i}{2m} \frac{k_\rho}{k^2} \epsilon^{\mu\nu\rho} \Pi_{\mu\nu}(k) \tag{9}$$

with

$$\Pi^\mu_\mu(k) \equiv g^{\mu\nu} \Pi_{\mu\nu}(k) \tag{10}$$

3. EVALUATION OF $\Pi_{\mu\nu}(k)$ IN THE HEISENBERG PICTURE

The vacuum polarization tensor in the HP is given by

$$\Pi_{\mu\nu}(k) = \frac{1}{2(2\pi)^2} \iint d^3p_1\, d^3p_2\, \delta(k - p_1 + p_2) \times \mathrm{Tr}\{\gamma_\mu(\not p_1 + m)\gamma_\nu(\not p_2 + m)\}[\Pi^- + \Pi^+] \tag{11}$$

where

$$\Pi^- \equiv \delta(m^2 - p_1^2)\left[PV\frac{1}{m^2 - p_2^2} - i\pi\varepsilon(p_2)\delta(m^2 - p_2^2)\right] \tag{12}$$

$$\Pi^+ \equiv \delta(m^2 - p_2^2)\left[PV\frac{1}{m^2 - p_1^2} + i\pi\varepsilon(p_1)\delta(m^2 - p_1^2)\right] \tag{13}$$

In the above expressions, PV denotes the principal value, while $\varepsilon(p)$ stands for the sign function

$$\varepsilon(p) = \frac{p}{|p|} \tag{14}$$

For physical reasons, i.e., in order to check the mass generation for the photon field (Pimentel *et al.*, 1992), let us first calculate the antisymmetric part of $\Pi_{\mu\nu}(k)$. After performing the trace and the p_2 integration in equation (11), we substitute the resulting expression for $\Pi_{\mu\nu}(k)$ into equation (9). So, $\Pi^{(2)}(k^2)$ becomes

$$\Pi^{(2)}(k^2) = \mathscr{R}\mathrm{e}\,\Pi^{(2)} + i\,\mathscr{I}\mathrm{m}\,\Pi^{(2)} \tag{15}$$

with

$$\mathscr{R}\mathrm{e}\,\Pi^{(2)} \equiv -\frac{1}{(2\pi)^2}\int d^3p\left[\delta(m^2 - p^2)PV\frac{1}{m^2 - (p-k)^2} + \delta[m^2 - (p-k)^2]PV\frac{1}{m^2 - p^2}\right] \tag{16}$$

$$\mathscr{I}\mathrm{m}\,\Pi^{(2)} \equiv -\frac{1}{4\pi}\int d^3p\,\delta(m^2 - p^2)\,\delta[m^2 - (p-k)^2]\{\varepsilon(p) - \varepsilon(p-k)\} \tag{17}$$

Making use of representations

$$\delta(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega a} \tag{18}$$

$$PV\frac{1}{b} = \frac{1}{2i}\int_{-\infty}^{\infty} d\omega\,\frac{\omega}{|\omega|}\,e^{i\omega b} \tag{19}$$

it follows that

$$PV\left\{\frac{\delta(a)}{b} + \frac{\delta(b)}{a}\right\} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\,\omega\int_0^1 d\alpha\, e^{i\omega[\alpha a + (1-\alpha)b]} \tag{20}$$

From equation (20), after a shift $p \to p + (1-\alpha)k$, we can rewrite equation (16) for the real part of $\Pi^{(2)}(k^2)$ as

$$\mathscr{R}\mathrm{e}\,\Pi^{(2)} = \frac{i}{(2\pi)^3}\int_0^1 d\alpha\int_{-\infty}^{\infty} d\omega\,\omega\int d^3p\, e^{i\omega(M^2 - p^2)} \tag{21}$$

where

$$M^2 \equiv m^2 - \alpha(1-\alpha)k^2 \tag{22}$$

We can easily compute the p and ω integrals in equation (21) with the aid of representations

$$\frac{1}{(A+i\epsilon)^{1+\lambda}} = \frac{(-i)^{1+\lambda}}{\Gamma(1+\lambda)} \int_0^\infty dx\, x^\lambda\, e^{ix(A+i\epsilon)} \tag{23}$$

$$\frac{1}{(A-i\epsilon)^{1+\lambda}} = \frac{(i)^{1+\lambda}}{\Gamma(1+\lambda)} \int_0^\infty dx\, x^\lambda\, e^{-ix(A-i\epsilon)} \tag{24}$$

Hence, substituting equation (22) for M^2, we finally obtain

$$\mathscr{R}\mathrm{e}\, \Pi^{(2)} = -\frac{1}{4\pi} \int_0^1 d\alpha \frac{1}{[m^2 - \alpha(1-\alpha)k^2]^{1/2}} \tag{25}$$

The imaginary part of $\Pi^{(2)}(k^2)$ is given by equation (17), i.e.,

$$\mathscr{I}\mathrm{m}\, \Pi^{(2)} = -\frac{1}{4\pi} \int d^2\mathbf{p} \int dp_0\, \delta(m^2 - p_0^2 + \mathbf{p}^2) \times \delta(m^2 - p_0^2 + \mathbf{p}^2 + 2p_0 k_0 - 2\mathbf{p}\cdot\mathbf{k} - k^2)\left(\frac{p_0}{|p_0|} - \frac{p_0 - k_0}{|p_0 - k_0|}\right) \tag{26}$$

Integration with respect to p_0 in the above expression in a coordinate system where $\mathbf{k} = 0$ leads to

$$\mathscr{I}\mathrm{m}\, \Pi^{(2)} = -\frac{1}{16} \int_0^\infty da \frac{1}{(m^2 + a)^{1/2}}\, \delta\left(a - \frac{k_0^2}{4} + m^2\right)\Delta \tag{27}$$

where

$$\Delta \equiv \frac{(m^2+a)^{1/2} + k_0}{|(m^2+a)^{1/2} + k_0|} - \frac{(m^2+a)^{1/2} - k_0}{|(m^2+a)^{1/2} - k_0|} \tag{28}$$

We notice that for the physical sector $k^2 < 4m^2$

$$\mathscr{I}\mathrm{m}\, \Pi^{(2)} = 0 \tag{29}$$

From the covariance of $\Pi_{\mu\nu}(k)$, equation (29) remains valid in an arbitrary frame of reference.

We thus verify from equations (25) and (29) that for $k^2 = 0$,

$$\Pi^2(0) = -\frac{1}{4\pi m} \tag{30}$$

We emphasize that in *all* these calculational steps, the $\Pi^{(2)}(k^2)$ remains *finite*. Hence, the conclusion we arrive at with the use of HP is that the $\Pi^{(2)}(k^2)$ term *does not* need to be regularized at any step in its calculation, since it is finite throughout. And this is our key result: Even though we do not need to use any regularization scheme to compute $\Pi^{(2)}(k^2)$, it does not vanish in the limit $k^2 \to 0$, so that it definitely contributes to the dislocation of the pole of the photon propagator, i.e., *it does generate photon mass at the one-loop quantum level.*

On the other hand, $\Pi^{(1)}(k^2)$ diverges. In other words, in the HP, *all* the divergence of the vacuum polarization tensor, equation (6), is carried by the $\Pi^{(1)}(k^2)$ term. Therefore, for this term we necessarily *must make use of a regularization procedure* to deal with it. Such can be chosen among several known gauge-invariant regularization procedures, e.g., the one we used in our previous work (Pimentel *et al.*, 1992). The final result for $\Pi^{(1)}(k^2)$, after regularization, is

$$\Pi^{(1)}(k^2) = \mathscr{R}\mathrm{e}\,\Pi^{(1)}(k^2) = \frac{k^2}{2\pi}\int_0^1 d\alpha\,\alpha(1-\alpha)\frac{1}{[m^2-\alpha(1-\alpha)k^2]^{1/2}} \tag{31}$$

Therefore, at $k^2 = 0$ we have

$$\Pi^{(1)}(0) = 0 \tag{32}$$

From the above results, equations (30) and (32), one can see clearly that the mass generation for the photon field just comes from the $\Pi^{(2)}(k^2)$ term—and solely from it. This is the reason why we omit most of the details for the calculation of $\Pi^{(1)}(k^2)$.

4. DISCUSSION

We would like to make some clarifying comments in addressing some of the issues pertaining to the Abelian gauge theories in three dimensions.

The antisymmetric piece of the vacuum polarization tensor

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi^{(1)}(k^2) - im\epsilon_{\mu\nu\alpha}k^\alpha\Pi^{(2)}(k^2)$$

when properly taken into account, gives rise to a corrected gauge boson propagator

$$D_{\mu\nu}(k) = \frac{-i}{k^2-\Pi(k^2)}\left\{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - im\frac{\Pi^{(2)}(k^2)}{1-\Pi^{(1)}(k^2)/k^2}\epsilon_{\mu\nu\alpha}\frac{k^\alpha}{k^2}\right\} - ia\frac{k_\mu k_\nu}{k^4 + i0k^2}$$

where

$$\Pi(k^2) = \Pi^{(1)}(k^2) + \frac{m^2\{\Pi^{(2)}(k^2)\}^2}{1-\Pi^{(1)}(k^2)/k^2}$$

a is the gauge fixing parameter. When

$$\Pi^{(1)}(0) = 0$$

and

$$\Pi^{(2)}(0) \neq 0$$

it can be regarded as the propagator for a gauge boson of square *topological* mass

$$\mu^2 \equiv \Pi(0) = \frac{(e^2/4\pi)^2}{1 - e^2/48\pi m}$$

Thus, the radiatively induced mass for the gauge boson has a topological origin, and corresponds to the addition of a topological mass counterterm

$$\frac{\mu}{4} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha \tag{33}$$

to the unrenormalized Lagrangian density, also known as the Chern–Simons term.

Under gauge transformations,

$$A_\mu \to A_\mu + \partial_\mu \Omega$$
$$\psi \to e^{ie\Omega}\psi$$

the renormalized Lagrangian density of spinor quantum electrodynamics in three dimensions (QED$_3$), including the Chern–Simons term (33), changes by a total derivative;

$$\mathscr{L} \to \mathscr{L} + \partial_\alpha \left\{ \frac{\mu}{4} \epsilon^{\alpha\mu\nu} F_{\mu\nu} \Omega \right\}$$

So, the Heisenberg equations of motion remain invariant.

5. CONCLUSION

We have considered an alternative approach to the vacuum polarization process in QED$_3$, where the interacting field operators satisfy Heisenberg equations of motion. As a consequence of equation (30), mass is generated for the photon at this level of radiative correction, since the antisymmetric part of the vacuum polarization tensor modifies the original pole in the photon propagator at $k^2 = 0$ (Deser *et al.*, 1982). This result is in agreement with the one obtained in the IP, where the corresponding Feynman integral is regularized by means of the analytic regularization method, constrained to preserve gauge invariance (Pimentel *et al.*, 1992).

It must be emphasized that while $\Pi^{(1)}(k^2)$ diverges in the ultraviolet (UV) region, $\Pi^{(2)}(k^2)$ remains finite and well defined along the intermediate steps of the present calculation in the HP. In comparison, in the IP, the Feynman amplitude for the photon self-energy is also UV-divergent. However, $\Pi^{(2)}(0)$ may vanish or not, depending on the choice of regularization (Deser *et al.*, 1982; Martin, 1990; Alverez-Gaumé *et al.*, 1990; Pimentel *et al.*, 1992) for the amplitude as a whole. This ambiguity in the physical result is then removed if we work in the HP, imposing current conservation.

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